Stability of Micromagnetic Algorithms with Thermal Noise

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Outline

- Thermal noise is important in micromagnetic calculation. for example, in the study of thermal stability of recording media.
- First-order algorithms are the simplest way to treat noise.
  Micromagnetic calculations without noise frequently use high-order integration schemes, because they allow the use of larger time steps and may be more stable. However, this is much more difficult in the context of stochastic differential equations (noise) -- the most straightforward algorithms remain first order.
- A new first-order quaternion algorithm
  is relatively stable compared to the usual Euler algorithm.
Landau-Lifshitz Equation

• The basic equation used here is the Landau-Lifshitz equation:

\[
\frac{dM}{dt} = -\gamma M \times H - \frac{\gamma \alpha}{M_s} M \times (M \times H)
\]

Here \(M\) is the magnetization vector, \(\gamma\) is the gyromagnetic ratio, \(\alpha\) is the LL damping coefficient. The magnetic field \(H\) includes external, exchange and anisotropy terms (a random term is not relevant to stability).

• The total field in a computational cell can be written as:

\[
H(r) = H_{\text{ext}} + \sum_{\delta} J M_r (r + \delta) + \frac{H_k}{M_s} (M(r) \cdot \hat{e}) \hat{e}
\]

Here \(\delta\) is the displacement to the six neighbors of a cell, \(J\) is the exchange integral, \(\hat{e}\) is a unit vector along the easy axis, \(H_k\) is the anisotropy field.
Exact Spin Wave Solutions

We will evaluate analytically the errors and instability of two first-order algorithms in the linear regime where we can describe the system in terms of its normal modes.

Consider a system with one spin wave, of wavevector $k$ and amplitude $M_k$. With circular polarization, the magnetization precesses about the external field (taken in the $z$ direction) and the precession angle $\theta = k \cdot r - \omega t$ varies with position $r$. One can show that this precession is a normal mode of the system (assumed infinite and periodic, for simplicity). This spin wave has frequency

$$\omega = \gamma \left[ H_{ext} + H_k \frac{M_z}{M_s} + JM_z B(k) \right]$$

Notice that external, anisotropy and exchange contributions to the spin wave frequency have been separated. The exchange contribution depends on the wavevector through the dimensionless function

$$B(k) = 6 - \sum_{\delta} e^{i k \cdot \delta} = 6 - 2[\cos(k_x a) + \cos(k_y a) + \cos(k_z a)]$$
Algorithms

Euler

\[ M(t+\Delta t) = M(t) + \left[ \frac{dM}{dt} \right] \Delta t \]

\[ \gamma H M \Delta t \]

H_{ext} only

(no exchange)

Top view

(with exchange)

Quaternion

\[ M(t+\Delta t) \]

[computed by rotating \( M(t) \)]

\[ \gamma H \Delta t \]

\[ \text{(bama.ua.edu/~visscher/mumag/quat.pdf)} \]
Lyapunov Exponents

Both the Euler and quaternion algorithms are numerically unstable, in that a spin wave amplitude will slowly grow, proportionately to $\exp(\lambda_{\text{num}} t)$. The growth rate $\lambda_{\text{num}}$ is called the numerical Lyapunov exponent and can be made as small as desired by decreasing the time step $\Delta t$. We have computed it as

$$\lambda_{\text{num}}(k,\Delta t) = B(k)^2 [\gamma J M_s]^2 \Delta t / 2 \quad \text{(Euler)}$$
$$\lambda_{\text{num}}(k,\Delta t) = [B(k)-6]B(k)[\gamma J M_s]^2 \Delta t / 2 \quad \text{(Quaternion)}$$

It is apparent from the graph below that the quaternion algorithm is about twice as stable as the Euler algorithm.
Stabilization by Damping

The numerical instability is not as serious as it might seem, because all real systems have some damping, which can stabilize the system by making a negative contribution $-\alpha \omega$ ($\alpha$ is the LL damping coefficient) to the net growth rate $\lambda_{\text{net}}(k, \Delta t) = \lambda_{\text{num}}(k, \Delta t) - \alpha \omega$:

**Numerical & damping contributions**

**Net growth rates**
Choosing a stable time step

Each algorithm is absolutely stable if the net growth rate
\[ \lambda_{\text{net}}(k, \Delta t) = \lambda_{\text{num}}(k, \Delta t) - \alpha \omega \]
is negative for all wave vectors, and unstable if \( \lambda_{\text{net}} \) is positive for any wave vector. The most unstable \( k \) is at the Brillouin zone boundary, where \( B(k) = 12 \). Here the exchange term usually dominates, so the frequency \( \omega = \gamma J M_s B(k) \).

For the Euler algorithm, for example, stability then requires
\[ \Delta t < \frac{\alpha}{6 \gamma J M_s} \]
which we define as \( \Delta t_{\text{crit}} \), the critical time step above which the system will be unstable. The graph shows the net growth rate for stable, unstable, and critical cases.
Conclusions

• We have calculated and verified numerically the linear stability growth rates for Euler and quaternion methods, both of them are first-order algorithms appropriate for micromagnetic systems with thermal noise.

• We find that the quaternion method is twice as stable as the standard Euler algorithm -- the quaternion method allows twice as long a time step to be used for a given Landau-Lifshitz damping rate $\alpha$. 